Effective Boundary Field Theory for a Josephson Junction Chain with a Weak Link

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Abstract

We show that a finite Josephson Junction (JJ) chain, ending with two bulk superconductors, and with a weak link at its center, may be regarded as a condensed matter realization of a two-boundary Sine-Gordon model. Computing the partition function yields a remarkable analytic expression for the DC Josephson current as a function of the phase difference across the chain. We show that, in a suitable range of the chain parameters, there is a crossover of the DC Josephson current from a sinusoidal to a sawtooth behavior, which signals a transition from a regime where the boundary term is an irrelevant operator to a regime where it becomes relevant.

Key words: Boundary conformal field theories, Josephson junction arrays and wire networks
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1 Introduction

In this paper, we analyze a superconducting 1+1-dimensional system, defined on a finite interval of length L. If the bulk is described by a massless theory, and conformal boundary conditions are chosen, one could understand the properties of the model, using the formalism of boundary conformal field theories [1]. If one deviates from this situation by either adding an interaction in the bulk, or at the boundary, or both, the behavior of the system becomes much more interesting, since it involves crossovers depending on the bulk and boundary energy scales, as well as on the finite size L.

In the sequel, we shall show that a JJ-chain with a weak link at its center and ending with two bulk superconductors at fixed phase difference φ, is the prototype of a condensed matter realization of a two-boundary Sine-Gordon model [2], whose Hamiltonian is given by

$$H = \frac{1}{4\pi} \int\limits_0^L dx \left[\frac{1}{v} \left(\frac{\partial \Phi}{\partial t} \right)^2 + v \left(\frac{\partial \Phi}{\partial x} \right)^2 \right] - \Delta_L \cos \left[\frac{\sqrt{g}}{2} \Phi(0) \right] - \Delta_R \cos \left[\frac{\sqrt{g}\Phi(L) - \varphi}{2} \right] \quad .(1)$$

Boundary field theories appear to be relevant in several different contexts. In condensed matter physics, they are mostly generalizations of quantum impurity models, which may be described by using the Tomonaga-Luttinger Liquid (TLL)-paradigm [3]; for instance, boundary interactions appear in the analysis of the Kondo problem [4], in the study of a one-dimensional conductor in presence of an impurity [5], and in the derivation of the tunneling between edge states of a Hall bar [6]. The TLL paradigm shows that many interactions are simply diagonalizable in the basis of appropriate collective bosonic modes, and that non diagonalizable interaction usually correspond to exactly solvable Hamiltonians, such as Sine-Gordon models [7,8]. Recently, boundary field theories have been investigated in the context of string theories. For instance, in studying tachyon instabilities [9], one is faced with the fact that the space of interacting string theory [10] may be mapped onto the space of boundary perturbations of conformal theories [11], and that the renormalization group flow determined by boundary perturbations may be identified with tachyon condensation [12]. Affleck and Ludwig [13] showed that the boundary entropy g is decreasing along the renormalization group trajectories, triggered by the boundary interaction.

In a inspiring paper [14], Glazman and Larkin analyzed the quantum phase diagram of a JJ-chain in the $V_g - J$-plane, where V_g is an external gate voltage applied to each junction, while J is the Josephson coupling between neighboring grains. They found evidence that this system undergoes a phase transition between an attractive TLL phase, with $g < 1$, and a repulsive TLL phase, with $g > 1$. While the former phase is the one-dimensional analog of the superconducting phase [15], the repulsive Tomonaga-Luttinger phase is peculiar of a one-dimensional system [3]. To be self-contained, here we shall provide a detailed field-theoretical description of the one-dimensional infinite chain analyzed in Ref.[14]: our rather pedagogical derivation evidences how the one-dimensional JJ-chain may be described in terms of interacting 1+1-dimensional chiral fermions and how, using the TLL paradigm, the interaction is exactly diagonalized in a pertinent basis. The TLL-g parameter [16] is crucial for the analysis of the phase diagram. Indeed, while for $g < 1$ the system supports an attractive TLL phase (superconducting), for $g > 1$ the JJ-chain is described by a repulsive TLL phase [14]. The $g = 1$-line corresponds to

a noninteracting TLL model. All this features may be quantitavely derived within the framework of the bosonized 1+1-dimensional TLL-model [1].

Using the TLL paradigm, we show that a finite JJ-chain with a weak link at its center is mapped onto a two-boundary Sine-Gordon model, with fixed Dirichlet boundary conditions at the outer boundary, and with dynamical boundary conditions at the inner boundary. To study the effects of the interaction at the inner boundary, we perform a Renormalization Group (RG) analysis, to derive how the effective parameters of the system scale as a function of the size of the chain L. We find that in the repulsive TLL-phase ($g > 1$) the boundary term is perturbative for any size L. At variance, in the attractive TLL-phase ($g < 1$), we find that there is an RG-invariant length, L^*, such that, for $L < L^*$ the boundary term is perturbative, while for $L \geq L^*$ it becomes nonperturbative. As for the models analyzed in [2], the crossover from the perturbative to the nonperturbative regime is evidenced by a change of the DC Josephson current (as a function of the phase difference at the bulk superconductors φ) from a sinusoidal to a sawtooth behavior.

The paper is organized as follows:

- In Section II we analyze the infinite one-dimensional JJ-array described in Ref.[14] and provide a detailed derivation of the mapping of this chain onto the anisotropic (XXZ) spin 1/2 model;
- In Section III we construct the effective field theory for the equivalent XXZ chain. We bosonize the theory and identify the various parameters of the continuum model in terms of the microscopic parameters of the lattice Hamiltonian;
- In Section IV, using the TLL-paradigm, we derive the phase diagram of the JJ-chain;
- In Section V we show that the effective field theory for the JJ-chain with a weak link and ending with two bulk superconductors is indeed the two-boundary Sine-Gordon model;
- In Section VI, using the Coulomb Gas Renormalization Group approach, we provide a careful estimate of the partition function of the two-boundary Sine-Gordon model for any value of g. We then derive the DC Josephson current across the chain, as a function of φ, at both fixed points and explicitly show the existence of a crossover from a sinusoidal to a sawtooth behavior;
- Section VII is devoted to a discussion of our results.

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[1] Notice that here the TLL-g parameter is the inverse of the parameter used in Ref.[14].

2 Mapping of the one-dimensional JJ-chain onto the XXZ spin-1/2 model

The simplest model Hamiltonian describing a one-dimensional JJ-chain is given by:

$$H = H_C + H_J \equiv \frac{E_C}{2} \sum_{j=1}^{L/a} \left(-i\frac{\partial}{\partial \phi_j} - \frac{N}{2} \right)^2 - J \sum_{j=1}^{L/a} \cos(\phi_j - \phi_{j+1}) \quad . \qquad (2)$$

In Eq.(2) $-i\frac{\partial}{\partial \phi_j}$ is the operator representing the number of Cooper pairs at site j in the phase representation and, thus, it takes only integer eigenvalues, n_j, E_C is the charging energy of a grain, J is the Josephson coupling energy and N accounts for the influence of a gate voltage, since $eN \propto V_g$. The sum over j ranges over the (L/a) sites, with L being the length of the chain, and a being the intergrain distance; imposing periodic boundary conditions amounts to fix $\phi_{L/a+j} = \phi_j$. For $J/E_C \rightarrow 0$, the chain is an insulator for almost any value of N, since it costs an energy $\sim E_C$, to change the number of pairs at any grain: E_C measures, then, the insulating gap.

When $N = n + 1/2 + h$, with integer n and $h \ll 1$, the two states $n_j = n$ and $n_j = n+1$ become almost degenerate in energy, even for large E_C; in this limit one may restrict the set of physical states to the Fock space F spanned by the $2^{L/a}$ states

$$|\{n\}\rangle = \prod_{j=1}^{L/a} |n_j\rangle \quad ; \; n_j = n, n+1 \quad .$$

The Josephson coupling lifts the degeneracy between n and $n+1$, since H_J may be represented as

$$H_J = -\frac{J}{2}[e^{i\phi_{j+1}} e^{-i\phi_j} + e^{i\phi_j} e^{-i\phi_{j+1}}] \quad , \qquad (3)$$

with the operator $e^{i\phi_j}$ $(e^{-i\phi_j})$ raising (lowering) the charge n_j by 1.

Resorting to the a well known procedure [17], one may easily construct the effective Hamiltonian H_{eff}, describing the JJ-chain on the reduced space F [14]. Let P be the projector onto F and $P_\perp$ be the projector onto the subspace $F_\perp$, to $O(J^2/E_C)$, H_{eff} takes the form:

$$H_{\text{eff}} = P(H_J + H_C)P + P\left[\frac{H_J P_\perp H_J}{-\frac{9}{16}E_C}\right]P \quad . \qquad (4)$$

When restricted to F, the operators $e^{\pm i\phi_j}$ and $-i\frac{\partial}{\partial \phi_j}$ may be represented with the spin-1/2 operators $S_j^\pm$ and S_j^z, as

$$Pe^{\pm i\phi_j}P = S_j^\pm \quad ; \quad P\left(-i\frac{\partial}{\partial \phi_j} - n - \frac{1}{2}\right)P = S_j^z \quad . \tag{5}$$

¿From Eq.(5), one immediately sees that a charge n $(n+1)$-state corresponds to a spin-(-1/2) (1/2)- eigenstate of S_j^z, and that, to $O(J^2/E_C)$, H_{eff} is given by

$$H_{\text{eff}} = -\frac{J}{2}\sum_{j=1}^{L/a}[S_j^+S_{j+1}^- + S_{j+1}^+S_j^-] - E_C h \sum_{j=1}^{L/a} S_j^z - \frac{3}{16}\frac{J^2}{E_C}\sum_{j=1}^{L/a} S_j^z S_{j+1}^z \quad . \tag{6}$$

To account for the contributions coming from intergrain capacitances, it is sufficient to retain only the nearest-neighbor terms [14], since next-to-nearest neighbor hopping terms would give rise to irrelevant operators, and to add to the Hamiltonian (6) the term $E_Z \sum_{j=1}^{L/a} S_j^z S_{j+1}^z$ $(E_Z > 0)$ [14]. Thus, the system is usefully described [14] by

$$H_{\text{Eff}} = -\frac{J}{2}\sum_{j=1}^{L/a}[S_j^+S_{j+1}^- + S_{j+1}^+S_j^-] - H\sum_{j=1}^{L/a} S_j^z + \Delta \sum_{j=1}^{L/a} S_j^z S_{j+1}^z \quad , \tag{7}$$

with $H = E_C h$ and $\Delta = E_Z - \frac{3}{16}\frac{J^2}{E_C}$.

Eq.(7) is the Hamiltonian for a spin-1/2 XXZ-chain in an external magnetic field H. The anisotropy parameter Δ may take positive, as well as negative values, depending on the constructive parameters of the JJ-chain. As elucidated in the following Sections, the sign of Δ is crucial for the emergence of a Repulsive Tomonaga-Luttinger (RTL) phase in a JJ-chain.

3 Continuum field theory and bosonization of the XXZ-chain

Using the standard bosonization technique [16], we derive in this Section the effective low-energy long-wavelength field theory associated to the Hamiltonian (7). For this purpose, one starts to write the spin operators S_j^a in terms of Jordan-Wigner (JW) spinless lattice fermions a_j [18], obeying standard anticommutation relations:

$$\{a_j, a_k^\dagger\} = \delta_{jk} \quad . \tag{8}$$

The JW transformation amounts to define:

$$S_j^+ \equiv a_j^\dagger \exp[i\pi \sum_{l=1}^{j-1} a_l^\dagger a_l] \quad ; \quad S_j^z \equiv a_j^\dagger a_j - \frac{1}{2} \quad ; \quad S_j^- \equiv \exp[-i\pi \sum_{l=1}^{j-1} a_l^\dagger a_l] a_j \quad , (9)$$

which, in turn, implies:

$$[S_j^a, S_i^b] = i\delta_{ji}\epsilon^{abc} S_j^c \quad . \tag{10}$$

¿From Eqs.(9), the fermionic effective Hamiltonian may be written as:

$$H_{\text{Eff}}^f \equiv H_K + H_P + H \sum_{j=1}^{L/a} a_j^\dagger a_j =$$

$$-\frac{J}{2}\sum_{j=1}^{L/a}[a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j] + \Delta \sum_{j=1}^{L/a}(a_j^\dagger a_j - \frac{1}{2})(a_{j+1}^\dagger a_{j+1} - \frac{1}{2}) + H \sum_{j=1}^{L/a} a_j^\dagger a_j .(11)$$

The hopping term H_K is readily diagonalized by resorting to the Fourier components of a_j, a_k,

$$a_j = \frac{1}{\sqrt{L/a}} \sum_k a_k e^{ik(ja)} \quad , \tag{12}$$

leading to:

$$H_K = \sum_k \epsilon(k) a_k^\dagger a_k \quad ; \quad (\epsilon(k) = -J\cos(ka) - H) \tag{13}$$

If $|H| < J$, the Fermi surface is disconnected and consists of two isolated points at $\pm k_F$, with $k_F = \arccos(H/J)$. Keeping only the excitations about the Fermi points with momenta k such that $|k \pm k_F| \leq \Lambda$, one obtains:

$$H_K \approx \sum_{|k-k_F|\leq\Lambda} \epsilon(k)a_k^\dagger a_k + \sum_{|k+k_F|\leq\Lambda} \epsilon(k)a_k^\dagger a_k \approx J\sin(ak_F) \sum_{|p|\leq\Lambda} \sin(pa)a_L^\dagger(p)a_L(p)$$

$$- J\sin(ak_F) \sum_{|p|\leq\Lambda} \sin(pa)a_R^\dagger(p)a_R(p) \quad , \tag{14}$$

with:

$$a_L(p) \equiv a_{p+k_F} \quad ; \quad a_R(p) \equiv a_{p-k_F} \; (|p| \leq \Lambda) \quad .$$

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For $\Lambda \ll k_F$, one may define the continuum chiral fields $\psi_{L/R}(x)$ as

$$\frac{a_j}{\sqrt{2\pi a}} \approx e^{ik_F x_j}\psi_L(x_j) + e^{-ik_F x_j}\psi_R(x_j) \quad , \tag{15}$$

with $x_j = ja$; one gets, then

$$H_K \approx J\sin(k_F a) \sum_{|p| \leq \Lambda} p[a_L^\dagger(p)a_L(p) - a_R^\dagger(p)a_R(p)]$$

$$= -iv_F \int_0^L dx \left[\psi_L^\dagger(x)\frac{d\psi_L(x)}{dx} - \psi_R^\dagger(x)\frac{d\psi_R(x)}{dx} \right] \tag{16}$$

where the Fermi velocity is given by $v_F = 2\pi J \sin(ak_F)$.

Eq.(16) is, of course, the effective low-energy theory of the hopping Hamiltonian H_K; the cutoff Λ will be specified later.

The dynamics of the fermionic fields ψ_L and ψ_R in the Heisenberg representation, is described by

$$\psi_L(x,t) = \psi_L(x - v_F t) = \frac{1}{\sqrt{L}} \sum_p e^{ip(x-v_F t)}\psi_L(p)$$

$$\psi_R(x,t) = \psi_R(x + v_F t) = \frac{1}{\sqrt{L}} \sum_p e^{ip(x+v_F t)}\psi_R(p) \quad , \tag{17}$$

and the equal time anticommutation relations are given by

$$\{\psi_L(p), \psi_L^\dagger(p^{'})\} = \delta_{p,p'} \;\; ; \;\; \{\psi_R(p), \psi_R^\dagger(p^{'})\} = \delta_{p,p'} \;\; ; \;\; \{\psi_R(p), \psi_L^\dagger(p^{'})\} = 0 \quad .\tag{18}$$

Since $\psi_L^\dagger(p)$ with $p > 0$ creates positive energy left-handed states, while $\psi_R^\dagger(p)$ creates positive energy right-handed states if $p < 0$, the "Fermi sea" fermionic ground state is defined as

$$|\text{FS}\rangle = \prod_{p<0}[\psi_L^\dagger(p)\psi_R^\dagger(-p)]|0\rangle \;\; (\psi_L(p)|0\rangle = \psi_R(p)|0\rangle = 0) \quad . \tag{19}$$

Thus, by choosing $\Lambda = 1/(4a)$, one gets:

$$\langle\text{FS}|2\pi a[e^{-2ik_F x_j}\psi_L^\dagger(x_j)\psi_R(x_j) + e^{2ik_F x_j}\psi_R^\dagger(x_j)\psi_L(x_j)]|\text{FS}\rangle = 0 \tag{20}$$

and:

$$\langle \mathrm{FS}|a[\psi_L^\dagger(x_j)\psi_L(x_j) + \psi_R^\dagger(x_j)\psi_R(x_j)]|\mathrm{FS}\rangle = \frac{1}{2} \quad ; \tag{21}$$

thus, S_j^z is normal ordered respect to $|\mathrm{FS}\rangle$, i.e.,

$$S_j^z = 2\pi a[: \psi_L^\dagger(x_j)\psi_L(x_j) : + : \psi_R^\dagger(x_j)\psi_R(x_j) :]+$$

$$2\pi a[: \psi_L^\dagger(x_j)\psi_R(x_j) : e^{-2ik_F x_j} + : \psi_R^\dagger(x_j)\psi_L(x_j) : e^{2ik_F x_j}] \tag{22}$$

where :: denotes normal ordering.

Using fermionic coordinates, one should now evaluate the Ising-Néel interaction H_P as

$$H_P \equiv \Delta \sum_{j=1}^N S_j^z S_{j+1}^z \equiv \Delta \sum_{j=1}^{L/a} (a_j^\dagger a_j - \frac{1}{2})(a_{j+1}^\dagger a_{j+1} - \frac{1}{2}) \approx$$

$$(4\pi^2 a\Delta) \int_0^L dx_j \{[: \psi_L^\dagger(x_j)\psi_L(x_j) : + : \psi_R^\dagger(x_j)\psi_R(x_j) :$$

$$+e^{-2ik_F x_j}\psi_L^\dagger(x_j)\psi_R(x_j) + e^{2ik_F x_j}\psi_R^\dagger(x_j)\psi_L(x_j)] \times$$

$$[: \psi_L^\dagger(x_{j+1})\psi_L(x_{j+1}) : + : \psi_R^\dagger(x_{j+1})\psi_R(x_{j+1}) : +$$

$$e^{-2ik_F x_{j+1}}\psi_L^\dagger(x_{j+1})\psi_R(x_{j+1}) + e^{2ik_F x_{j+1}}\psi_R^\dagger(x_{j+1})\psi_L(x_{j+1})]\} = H_P^{(1)} + H_P^{(2)} \quad ,(23)$$

where

$$H_P^{(1)} = (4\pi^2 a\Delta) \int_0^L dx_j \;[: \psi_L^\dagger(x_j)\psi_L(x_j) :: \psi_L^\dagger(x_{j+1})\psi_L(x_{j+1}) :$$

$$+ : \psi_R^\dagger(x_j)\psi_R(x_j) :: \psi_R^\dagger(x_{j+1})\psi_R(x_{j+1}) :$$

$$+ : \psi_L^\dagger(x_j)\psi_L(x_j) :: \psi_R^\dagger(x_{j+1})\psi_R(x_{j+1}) : + : \psi_R^\dagger(x_j)\psi_R(x_j) :: \psi_L^\dagger(x_{j+1})\psi_L(x_{j+1}) :] \quad ,(24)$$

and

$$H_P^{(2)} = (4\pi^2 a\Delta) \int_0^L dx_j \;[\psi_L^\dagger(x_j)\psi_R(x_j)e^{-2ik_F x_j} + \psi_R^\dagger(x_j)\psi_L(x_j)e^{2ik_F x_j}] \times$$

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$$[\psi_L^\dagger(x_{j+1})\psi_R(x_{j+1})e^{-2ik_Fx_{j+1}} + \psi_R^\dagger(x_{j+1})\psi_L(x_{j+1})e^{2ik_Fx_{j+1}}] \quad . \tag{25}$$

While evaluating $H_P^{(1)}$ is rather straightforward, since it contains only normal-ordered fermionic left- and right- densities, evaluating $H_P^{(2)}$ is a little bit more involved, due to "crossed" $L-R$-interaction. In fact, at any k_F, momentum conservation selects the pertinent contribution to Eq.(25), given by

$$H_P^{(2)} = (4\pi^2 a\Delta) \int_0^L dx_j \, [e^{2ik_Fa}\psi_L^\dagger(x_j)\psi_R(x_j)\psi_R^\dagger(x_{j+1})\psi_L(x_{j+1})+$$

$$e^{-2ik_Fa}\psi_R^\dagger(x_j)\psi_L(x_j)\psi_L^\dagger(x_{j+1})\psi_R(x_{j+1})] \quad , \tag{26}$$

where a possible "Umklapp" contribution, arising when $k_Fa \sim \pi/2$, has been neglected [2]. To normal order $H_P^{(2)}$, one may rewrite it as

$$H_P^{(2)} = -(4\pi^2 a\Delta)e^{2ik_Fa} \int_0^L dx \left[: \psi_L^\dagger(x)\psi_L(x+a) : + \frac{i}{2\pi a}\right]\left[: \psi_R^\dagger(x+a)\psi_R(x) : + \frac{i}{2\pi a}\right]$$

$$- (4\pi^2 a\Delta)e^{-2ik_Fa} \int_0^L dx \left[: \psi_L^\dagger(x+a)\psi_L(x) : - \frac{i}{2\pi a}\right]\left[: \psi_R^\dagger(x)\psi_R(x+a) : - \frac{i}{2\pi a}\right] \quad , \tag{27}$$

which, for $a \to 0$, becomes

$$H_P^{(2)} = -2(4\pi^2 a\Delta)\cos(2k_Fa) \int_0^L dx \; : \psi_L^\dagger(x)\psi_L(x) :: \psi_R^\dagger(x)\psi_R(x) :$$

$$+4\pi\Delta \sin(2k_Fa) \int_0^L dx \, [: \psi_L^\dagger(x)\psi_L(x) : + : \psi_R^\dagger(x)\psi_R(x) :]$$

$$- i4\pi a\Delta \cos(k_Fa) \int_0^L dx \left[\psi_L^\dagger(x)\frac{d\psi_L(x)}{dx} - \psi_R^\dagger(x)\frac{d\psi_R(x)}{dx}\right] \quad . \tag{28}$$

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[2] This is the case, for instance, of the "half filled" fermionic sea in zero chemical potential

The various terms in Eq.(28) may be interpreted as follows:

- A shift in the chemical potential:

$$4\pi\Delta\sin(2k_Fa)\int_0^L dx\,[:\psi_L^\dagger(x)\psi_L(x): + :\psi_R^\dagger(x)\psi_R(x):]\ ,$$

which is accounted for by simply redefining k_F through the equation

$$-(aJ)\cos(k_Fa) + 2\Delta\sin(2k_Fa) = H\ ; \tag{29}$$

- A $L-R$ interaction term:

$$-2(4\pi^2a\Delta)\cos(2k_Fa)\int_0^L dx\ :\psi_L^\dagger(x)\psi_L(x)::\psi_R^\dagger(x)\psi_R(x):\ ,$$

that adds up to a similar term coming from $H_P^{(1)}$, giving

$$2(4\pi^2a\Delta)[1-\cos(2k_Fa)]\int_0^L dx\ :\psi_L^\dagger(x)\psi_L(x)::\psi_R^\dagger(x)\psi_R(x):\ ; \tag{30}$$

- A renormalization of the Fermi velocity given by

$$-i4\pi a\Delta\cos(k_Fa)\int_0^L dx\ \left[\psi_L^\dagger(x)\frac{d\psi_L(x)}{dx} - \psi_R^\dagger(x)\frac{d\psi_R(x)}{dx}\right]\ . \tag{31}$$

Using the well-known bosonization rules (A.9), the fermionic Hamiltonian H_{Eff}^f may be written in bosonic coordinates as

$$H^b = \frac{v_F + g_2}{4\pi}\int_0^L dx\ \left[\left(\frac{\partial\phi_L}{\partial x}\right)^2 + \left(\frac{\partial\phi_R}{\partial x}\right)^2\right] + 2\frac{g_4}{4\pi}\int_0^L dx\ \left[\frac{\partial\phi_L}{\partial x}\frac{\partial\phi_R}{\partial x}\right]\ , \tag{32}$$

where $g_2 = g_4 = 4\pi(a\Delta)[1-\cos(2k_Fa)]$.

One may readily see that H^b corresponds to the Hamiltonian for a free, massless, real bosonic field Φ in 1+1 dimensions, which is described by the Hamiltonian

$$H[\Pi,\Phi] = \frac{v}{4\pi}\int_0^L dx\ \left[\frac{4\pi^2}{g}\Pi^2 + g\left(\frac{\partial\Phi}{\partial x}\right)^2\right] \tag{33}$$

where the momentum conjugate to Φ is $\Pi = (2\pi/g)\frac{\partial\Phi}{\partial t}$.

Upon defining two independent chiral fields, ϕ_L^g and ϕ_R^g, as

$$\frac{\partial \phi_L^g(x - vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[\frac{2\pi}{\sqrt{g}} \Pi + \sqrt{g} \frac{\partial \Phi}{\partial x} \right]$$

$$\frac{\partial \phi_R^g(x + vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[-\frac{2\pi}{\sqrt{g}} \Pi + \sqrt{g} \frac{\partial \Phi}{\partial x} \right] \quad , \tag{34}$$

one immediately sees that

$$H[\Pi, \Phi] \to H[\phi_L^g, \phi_R^g] = \frac{v}{4\pi} \int\limits_0^L dx \left[\left(\frac{\partial \phi_L^g}{\partial x} \right)^2 + \left(\frac{\partial \phi_R^g}{\partial x} \right)^2 \right] \quad , \tag{35}$$

which, when expressed in terms of $\phi_L^{g=1}$ and $\phi_R^{g=1}$, yields Eq.(32), provided that

$$v = \sqrt{(v_F + g_2)^2 - g_4^2} \quad ; \quad g = \sqrt{\frac{v_F + g_2 + g_4}{v_F + g_2 - g_4}} \quad . \tag{36}$$

Thus, the correlation functions of all the operators depending on ϕ_L and ϕ_R may be evaluated by the replacements

$$\phi_L - \phi_R = \sqrt{g}[\phi_L^g - \phi_R^g] \quad , \quad \phi_L + \phi_R = \sqrt{\frac{1}{g}}[\phi_L^g + \phi_R^g] \quad , \tag{37}$$

with ϕ_L^g, ϕ_R^g free, chiral bosonic fields.

4 Phase diagram of the JJ-chain

In Ref.[14], it has been evidenced that the phases allowed to a JJ-chain are:

(1) A "Mott insulating" (MI) phase;
(2) A "band isulating" (BI) phase;
(3) A Repulsive Tomonaga-Luttinger phase (RTL);
(4) A Superconducting (S), attractive Tomonaga-Luttinger phase.

Here, we shall determine the range of the JJ-chain parameters associated to each allowed phase in the $V_g - J$-plane and, using the TLL approach, we shall provide a careful derivation of the phase boundaries; of course, our results crucially depend on the approximations made in Section 2. Our subsequent

analysis is based on the bosonic, low-energy effective Hamiltonian given in Eq.(32).

To analyze the onset of the MI phase, one has to include also the Umklapp term H_P^u in Eq.(25), given by

$$H_P^u \approx -(a\Delta) \int_0^L dx_j \; [\psi_L^\dagger(x_j)\psi_L^\dagger(x_{j+1})\psi_R(x_j)\psi_R(x_{j+1}) + \psi_R^\dagger(x_j)\psi_R^\dagger(x_{j+1})\psi_L(x_j)\psi_L(x_{j+1})] \quad , (38)$$

whose bosonized version yields

$$H_P^u = -\left(2\frac{g_U}{L^2}\right) \int_0^L dx \; : \cos[2\sqrt{2}\Phi(x)] : \; ; \; g_U = a\Delta \left(\frac{2\pi a}{L}\right)^{\frac{4}{g}} \quad . \tag{39}$$

Eq.(39) and Eq.(33) yield the Hamiltonian for a 1+1-dimensional Sine-Gordon model, whose phase structure, as a function of the parameters g and g_U has been extensively studied [19]. There are two distinct regimes: if $g < 2$, the interaction is irrelevant and the theory is perturbative in g_U, while, if $g > 2$, the interaction is relevant. In the thermodynamic limit ($L \to \infty$) the system flows towards a strongly-coupled regime, where the Umklapp interaction is responsible for the creation of a gap in the excitation spectrum and for the onset of long range Ising-Néel order [7]. In the language of the JJ-chain, this corresponds to a checkboard charge ordered state with the charge at each grain either equal to n, or to $n + 1$: this is the MI-phase.

The MI charge-ordered region in the $V_g - J$ plane may be identified with the condition $g > 2$, which reads

$$4\sin(ak_F)\left(E_Z - \frac{3}{16}\frac{J^2}{E_C}\right) > \frac{3}{2}J \quad . \tag{40}$$

As $J = 0$, Eq.(40) is satisfied by any value of k_F (provided that there is a real solution to Eq.(29)) When there is no real solutions to Eq.(29), that is, for large enough $|H|$, the chain undergoes a phase transition towards the BI phase. This shows that, for $J = 0$, the only possible phases are either the BI phase, or the MI charge-ordered phase.

To see how the transition towards the BI phase extends for $J > 0$, one may start again from Eq.(29), describing the boounday of the BI phase. If $H > 0$,

there are no real solutions of Eq.(29) for

$$H - 2\Delta > J \quad \Rightarrow \quad H - 2E_Z - J + \frac{3}{8}\frac{J^2}{E_C} > 0 \quad . \tag{41}$$

As $H < 0$, on the other hand, there are no real solutions if

$$H + 2\Delta < -J \quad \Rightarrow \quad H + 2E_Z + J - \frac{3}{8}\frac{J^2}{E_C} < 0 \quad . \tag{42}$$

Eqs.(41,42) define two regions in the phase diagram corresponding to BI phases, since the density of charge-carrying states at the Fermi surface is 0.

Furthermore, as $J > 0$, Eq.(40) admits real solutions only if

$$\frac{\frac{3}{8}J}{\left(E_Z - \frac{3}{16}\frac{J^2}{E_C}\right)} < 1 \quad \Rightarrow \quad J < \sqrt{E_C^2 + \frac{16}{3}E_Z E_C} - E_C \quad , \tag{43}$$

which implies that the MI phase closes when $J = \sqrt{E_C^2 + \frac{16}{3}E_Z E_C} - E_C$. Since Δ changes sign for $J = J^* = \sqrt{\frac{16}{3}E_Z E_C}$, one finds that, as the MI phase closes, the Tomonaga-Luttinger liquid interaction is still repulsive (that is, $g > 1$).

The phase where, instead, the Tomonaga-Luttinger liquid is attractive (which is a necessary condition, to achieve superconducting correlations in the 1-d system) takes place for $J > J^*$, that is, for $g < 1$.

In Fig.1 we plot the phase diagram obtained using the TLL-approach. We observe that, due to the renormalization of the Fermi velocity, the line corresponding to $g = 1$ is a straight horizontal line: thus, as long as the TLL-description of the JJ-chain holds, one cannot push the system across this line by acting on the gate voltage V_g. We expect that this behavior is a byproduct of the approximations introduced in Section 2; higher order contributions to H_{eff} should strongly modify the line corresponding to $g = 1$.

5 Two-boundary Sine-Gordon-model description of a finite JJ-chain

In the following, we shall consider a one-dimensional JJ-chain with a weak link (i.e., a junction with a different nominal value of the Josephson coupling, E_W) at its center, whose position is set at $x = 0$, and ending with two bulk
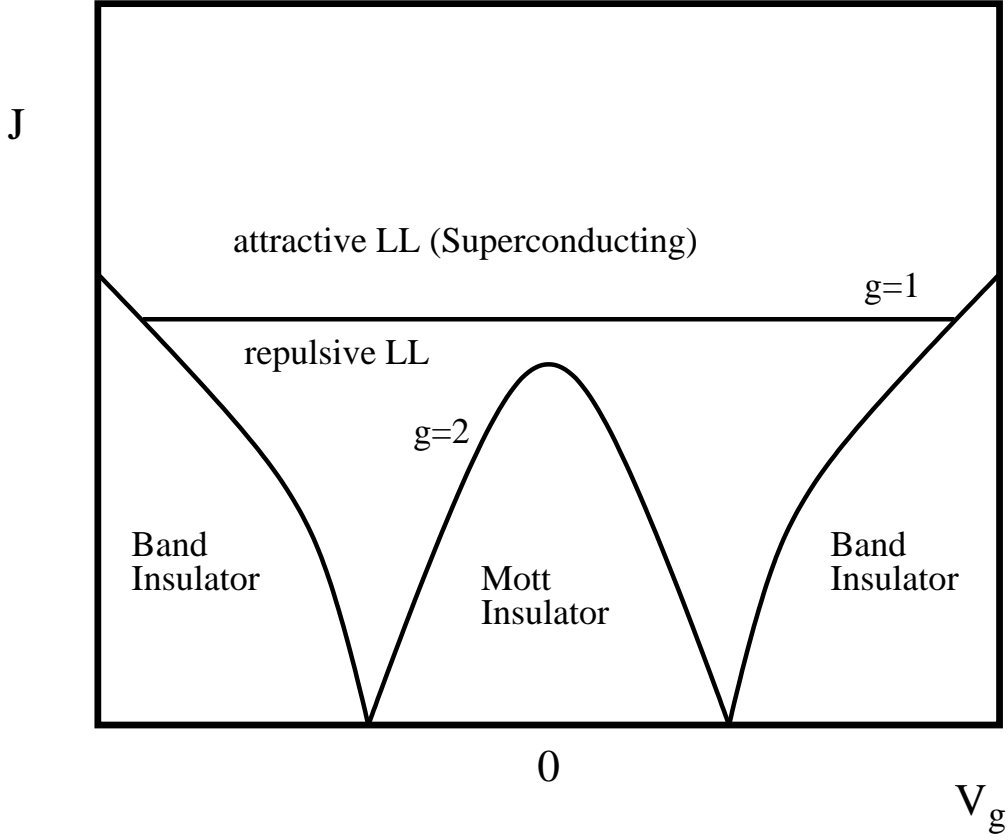
Fig. 1. Sketch of the phase diagram of the JJ-chain in the $V_g - J$ plane derived within TLL-approach, as discussed in Section IV.

superconductors, whose phase difference is held fixed at φ (i.e., $\varphi_R = -\varphi_L = \varphi/2$). Using the bosonization method, in this Section we show that this finite JJ-chain is pertinently described by a two-boundary Sine-Gordon model [2].

Upon introducing JW fermions on both sides of the weak link, one gets

$$S^+_{L/a,>} = [e^{i\pi \sum_{l=1}^{L/a-1} a_l^\dagger a_l}]a_{L/a}^\dagger = e^{i\varphi/2}$$

$$S^+_{-L/a,<} = [e^{i\pi \sum_{l=-1}^{-L/a-1} a_l^\dagger a_l}]a_{-L/a}^\dagger = e^{-i\varphi/2} \tag{44}$$

where the labels $_>$ $(_<)$ refer to observables at the right (left)-hand side of the weak link.

Using the long wavelength approximation, the fermionic string in the exponential of Eqs.(44), is easily evaluated as:

$$\pi \sum_{l=1}^{L/a-1} a_l^\dagger a_l = \frac{\pi L}{2a} + \int_{0^-}^{L-a} dx_l \, [: \psi_{L,>}^\dagger(x_l)\psi_{L,>}(x_l) : + : \psi_{R,>}^\dagger(x_l)\psi_{R,>}(x_l) :] =$$

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$$\frac{\pi L}{2a} + \frac{1}{2}[\phi_{L,>}(L-a) + \phi_{R,>}(L-a)] \quad , \tag{45}$$

which, in turn, implies:

$$S^+_{L/a,>} \approx \left(\frac{2\pi a}{L}\right)^{\frac{1}{2}} : e^{\frac{3i}{2}\phi_{L,>}(L)} : e^{\frac{i}{2}\phi_{R,>}(L)} : + \left(\frac{L}{2\pi a}\right)^{\frac{1}{2}} : e^{\frac{i}{2}\phi_{L,>}(L)} :: e^{-\frac{i}{2}\phi_{R,>}(L)} : \quad .(46)$$

and, by keeping only the leading contributions to Eq.(46) in the cutoff a, as $a \to 0$, one gets, for $x > 0$,

$$S^+_{L/a,>} \approx \left(\frac{L}{2\pi a}\right)^{\frac{1}{2}} : e^{\frac{i}{2}\phi_{L,>}(L)} :: e^{-\frac{i}{2}\phi_{R,>}(L)} := e^{\frac{i}{2}[\phi_{L,>}(L)-\phi_{R,>}(L)]} \quad . \tag{47}$$

Similarly, for $x < 0$, one obtains

$$S^+_{-L/a,<} = e^{\frac{i}{2}[\phi_{L,<}(-L)-\phi_{R,<}(-L)]} \quad . \tag{48}$$

The boundary condition at $x = 0$ is, instead, dynamical, since it depends on the strength of the weak link, E_W. In terms of the spin variables, the weak link interaction may be represented as a pointwise contact Hamiltonian given by

$$H_W = \frac{E_W}{2}[S^+_{0,<}S^-_{0,>} + S^+_{0,>}S^-_{0,<}] \quad . \tag{49}$$

Taking into account that $S^z_0 = \frac{1}{2}[a^\dagger_0 a_0 + a_0 a^\dagger_0] = \frac{a}{L}[: \psi^\dagger_L(0)\psi_L(0) + \psi^\dagger_R(0)\psi_R(0) :]$ and the requirement that $S^+_0 S^-_0 - S^-_0 S^+_0 = 2S^z_0$, for any value of g, the operators S^+_0 and S^-_0 are realized as:

$$S^+_0 = \frac{a}{L}\left(\frac{2\pi a}{L}\right)^{\frac{g}{2}} : e^{\frac{i}{2}[\phi_L(0)-\phi_R(0)]} :$$

$$S^-_0 = \frac{a}{L}\left(\frac{2\pi a}{L}\right)^{\frac{g}{2}} : e^{-\frac{i}{2}[\phi_L(0)-\phi_R(0)]} : \quad . \tag{50}$$

¿From Eqs.(50), the dependence of H_W on the bosonic coordinates is given by:

$$H_W = \frac{aE_W}{2L}\left(\frac{2\pi a}{L}\right)^g \left\{ : \exp\left[\frac{i}{2}[-\phi_{L,<}(0) + \phi_{R,<}(0) - \phi_{L,>}(0) + \phi_{R,>}(0)]\right] : +\text{h.c.} \right\} \quad .(51)$$

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Using Eq.(37), one immediately sees that the boundary interaction Hamiltonian at $x = 0$ takes the form

$$H_W = \frac{aE_W}{2L} \left(\frac{2\pi a}{L} \right)^g \left[: e^{i\sqrt{\frac{g}{2}}[\phi_{L,+}(0) - \phi_{R,+}(0)]} : + : e^{-i\sqrt{\frac{g}{2}}[\phi_{L,+}(0) - \phi_{R,+}(0)]} : \right] \quad (52)$$

where

$$\frac{\partial \phi_{L,+}(x - vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[\frac{2\pi}{\sqrt{g}} \Pi_+ + \sqrt{g} \frac{\partial \Phi_+}{\partial x} \right]$$

$$\frac{\partial \phi_{R,+}(x + vt)}{\partial x} = \frac{1}{\sqrt{2}} \left[-\frac{2\pi}{\sqrt{g}} \Pi_+ + \sqrt{g} \frac{\partial \Phi_+}{\partial x} \right] \quad , \quad (53)$$

with $\Phi_\pm(x,t) = \frac{1}{\sqrt{2}}[\Phi_>(x,t) \pm \Phi_<(-x,t)]$.

The boundary conditions at the bulk superconductors may be written as:

$$\sqrt{g}[\phi_{L,+}^g(L,t) - \phi_{R,+}^g(L,t)] = \varphi \pmod{2\pi k} \quad . \quad (54)$$

By inspection of Eqs.(52), one sees that the field $\Phi_-(x,t)$ fully decouples from the weak link dynamics. Furthermore, its boundary condition is $\Phi_-(L,t) = 0$ $\forall t$, thus implying that Φ_- is insensitive to variations in the phase difference between the bulk superconductors. As a result, the field Φ_- does not contribute to the dynamics of the JJ-network. Using Eq.(52), one gets that the pertinent effective Hamiltonian H_{JJ} is given by

$$H_{\text{JJ}} = \frac{v}{4\pi} \int_0^L dx \left[\left(\frac{\partial \phi_L}{\partial x} \right)^2 + \left(\frac{\partial \phi_R}{\partial x} \right)^2 \right] - \frac{aE_W}{L} \left(\frac{2\pi a}{L} \right)^g : \cos\left[\sqrt{\frac{g}{2}}(\phi_L(0) - \phi_R(0)) \right] : (55)$$

with $(\phi_{L,+}, \phi_{R,+}) \to (\phi_L, \phi_R)$, while the pertinent boundary condition is given by:

$$\sqrt{g}[\phi_L(L,t) - \phi_R(L,t)] = \varphi \pmod{2\pi k}. \quad (56)$$

The model described by H_{JJ}, supplemented with the boundary condition in Eq.(56), coincides with the two-boundary Sine-Gordon Hamiltonian introduced in Eq.(1), provided Φ in Eq.(1) is identified with $(\phi_L - \phi_R)/\sqrt{2}$, Δ_R is sent to ∞, and Δ_L is identified with $\frac{aE_W}{L} \left(\frac{2\pi a}{L} \right)^g$. Intuitively speaking, while

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the boundary condition at $x = L$ is always Dirichlet-like, at $x = 0$, since E_W is finite, the boundary condition is dynamical, and is given by:

$$\frac{aE_W}{L}\left(\frac{2\pi a}{L}\right)^g \sqrt{g}\sin\left[\sqrt{\frac{g}{2}}(\phi_L(0,t) - \phi_R(0,t))\right] + v\frac{\partial}{\partial x}[\phi_L(0,t) - \phi_R(0,t)] = 0 \quad .(57)$$

In particular, for small E_W, Eq.(57) provides Neumann-like boundary conditions for $\phi_L - \phi_R$ at $x = 0$, while it provides Dirichlet-like boundary conditions for large values of E_W.

For $g < 1$, the boundary term is a relevant operator and one may use the Renormalizion Group to describe how the renormalization of $\bar{E}_W$ affects the ground-state energy: as we shall evidence in section 6, there is a renormalization group invariant length scale L^* such that, for $L \geq L^*$ the JJ-chain behaves nonperturbatively in $\bar{E}_W$ while, for $L < L^*$, it behaves perturbatively. At variance, for $g > 1$, the JJ-chain is always perturbative in $\bar{E}_W$, since the boundary term is now an irrelevant operator. For $g = 1$, the boundary term is marginal and the bulk system is fully described by a pair of noninteracting chiral fermions and the partition function may be computed exactly [2].

To summarize, for $g > 1$ $\bar{E}_W$ always flows to 0, while, for $g < 1$, there is a characteristic " healing" length $L^* = L\left(\frac{J}{\bar{E}_W(\Lambda=1)}\right)^{\frac{1}{1-g}}$, separating a small-$\bar{E}_W$ perturbative regime from the nonperturbative one $\bar{E}_W \geq 1$. Similar features are exhibited by a superconducting loop closed by a Josephson junction of strength E_J, when E_J is regarded as an effective coupling strength [20].

In the next Section, we shall prove that the behavior of the DC Josephson current as a function of φ depends crucially on whether $\bar{E}_W$ flows to zero, or to large values. Namely, we shall find that, when $\bar{E}_W$ flows to zero, the DC Josephson current has a sinusoidal behavior, while, when $\bar{E}_W \sim 1$, one gets the sawtooth behavior.

6 Josephon current across the JJ-network with a weak link

In this Section, we shall compute the functional dependence of the DC Josephson current on the phase difference between the bulk superconductors, by evaluating the zero temperature canonical partition function $Z[E_W]$, from which the Josephson current may be evaluated as

$$I(\varphi) = \frac{2e}{c}\frac{\partial E_{JJ}[\varphi]}{\partial \varphi} \tag{58}$$

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with

$$E_{JJ}[\varphi] = -\lim_{\beta \to \infty} \frac{1}{\beta} \ln \left[\frac{Z[E_W]}{Z[0]} \right] \tag{59}$$

The partition function of the two-boundary Sine-Gordon model has been exactly computed for particular values of g [2]; due to our interest in providing an estimate of the Josephson current fon any value of g, we resort to an approximate computation based on the Coulomb Gas Renormalization Group scheme [21], described in detail in Appendix B. Our analysis shows that, while for Neumann boundary conditions ($E_W \to 0$), one gets $I(\varphi) \propto \sin(\varphi)$, for Dirichlet boundary conditions ($E_W \geq 1$) one gets a sawtooth dependence of $I(\varphi)$ on φ.

Case A: $E_W \to 0$

As $E_W \sim 0$, Eq.(57) provides Neumann-like boundary condition at $x = 0$,

$$\frac{\partial}{\partial x}[\phi_L(0 - vt) - \phi_R(0 + vt)] = 0 \ , \tag{60}$$

together with the Dirichlet-like boundary condition at $x = L$:

$$\phi_L(L - vt) - \phi_R(L + vt) = \frac{\varphi}{\sqrt{g}} \ .$$

Using the mode expansion given in Eq.(A.5), one sees that Eq.(60) is satisfied if

$$q_L - q_R = \frac{\varphi}{\sqrt{g}} \ ; \ p_L = p_R \equiv p$$

$$\phi_L(n)e^{ik_n L} = -\phi_R(-n)e^{-ik_n L} \equiv \alpha(n) \ ; \ (k_n = \frac{2\pi}{L}\left(n + \frac{1}{4}\right) \ , \ n \in Z) \tag{61}$$

which, in turn, implies

$$\phi_L(0 - vt) - \phi_R(0 + vt) = \frac{\varphi}{\sqrt{g}} + \frac{4\pi}{L} \sum_{n \neq 0} \frac{e^{-ik_n vt}}{k_n} \alpha(n) \equiv \chi(t) \ . \tag{62}$$

The relevant vertex operators are given by

$$: e^{\pm i \sqrt{g} \chi(t)} := e^{\pm i \varphi} e^{\pm i \sqrt{g} \chi_+(t)} e^{\pm i \sqrt{g} \chi_-(t)} \tag{63}$$

with

$$\chi_+(t) = \frac{4\pi}{L} \sum_{n<0} \frac{e^{-ik_n v t}}{k_n} \alpha(n) \ ; \ \chi_-(t) = \frac{4\pi}{L} \sum_{n>0} \frac{e^{-ik_n v t}}{k_n} \alpha(n) \ \ . \tag{64}$$

The partition function is given by

$$Z[E_W] = Z[E_W = 0] \left\langle T_\tau \left[\exp \left[\frac{aE_W}{L} \left(\frac{2\pi a}{L} \right)^g \int_0^\beta d\tau \ : \cos[\sqrt{\frac{g}{2}} \chi(\tau)] : \right] \right] \right\rangle \tag{65}$$

where $\tau = it$, T_τ is the (imaginary) time-ordered product, and the brackets $\langle \ldots \rangle$ mean that expectation values should be computed with respect to the ground state of the Hamiltonian in Eq.(35).

Using the expansion of Eq.(65) in a power series of E_W as

$$Z[E_W] = Z[E_W = 0] \sum_{j=0}^{\infty} \frac{(\bar{E}_W)^j}{2^j j!} \sum_{\alpha_j = \pm 1} \int_0^\beta \prod_{k=1}^j d\tau_k \left\langle T_\tau [\prod_{k=1}^j : e^{i\alpha_k \chi(\tau_k)} :] \right\rangle \tag{66}$$

with $\bar{E}_W = \frac{aE_W}{L} \left(\frac{2\pi a}{L} \right)^g$, and using the identity

$$\theta[\tau_1 - \tau_2][\chi_-(\tau_1), \chi_+(\tau_2)] = 4 \ln[1 - e^{-\frac{2\pi v (\tau_1 - \tau_2)}{L}}] \ \ , \tag{67}$$

one gets

$$\frac{Z[E_W]}{Z[E_W = 0]} = \sum_{j=0}^{\infty} \frac{(\bar{E}_W)^j}{2^j} \sum_{\alpha_j = \pm 1} \int_0^\beta d\tau_1 \int_0^{\tau_1 - a/v} d\tau_2 \ldots \int_0^{\tau_{j-1} - a/v} d\tau_j \times$$

$$e^{i \frac{\varphi}{\sqrt{g}} \sum_{k=1}^j \alpha_k} \prod_{u<r=1}^{2j} [1 - e^{\frac{-2\pi v}{L} |\tau_u - \tau_r|}]^{2g \alpha_u \alpha_r} \ \ , \tag{68}$$

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since Wick's theorem implies [3]

$$\langle T_\tau [\prod_{k=1}^{2j} : e^{i\sqrt{\frac{g}{2}}\alpha_k \chi(\tau_k)} :] \rangle = e^{i\frac{\varphi}{\sqrt{g}}\sum_{k=1}^{j}\alpha_k} \prod_{u<r=1}^{2j} [1 - e^{\frac{-2\pi v}{L}|\tau_u - \tau_r|}]^{2g\alpha_u\alpha_r} \quad . \quad (69)$$

To analyze the short-distance divergences in Eq.(68), one has to rescale the cutoff $a \to a/\Lambda$, $\Lambda > 1$, in order to approximate $\prod_{u<r=1}^{2j}[1 - e^{\frac{-2\pi v}{L}|\tau_u - \tau_r|}]^{2g\alpha_u\alpha_r}$ with $\prod_{u<r=1}^{2j} \left| \frac{2\pi v}{L}(\tau_u - \tau_r) \right|^{2g\alpha_u\alpha_r}$. As a result, one obtains

$$\int_0^\beta d\tau_1 \int_0^{\tau_1 - a/\Lambda v} d\tau_2 \ldots \int_0^{\tau_{j-1} - a/\Lambda v} d\tau_j e^{i\frac{\varphi}{\sqrt{g}}\sum_{k=1}^{j}\alpha_k} \prod_{u<r=1}^{2j} \left| \frac{2\pi v}{L}(\tau_u - \tau_r) \right|^{2g\alpha_u\alpha_r} =$$

$$\Lambda^{g\sum_{u\neq r}\alpha_u\alpha_r} \int_0^\beta d\tau_1 \int_0^{\tau_1 - a/v} d\tau_2 \ldots \int_0^{\tau_{j-1} - a/\,v} d\tau_j e^{i\frac{\varphi}{\sqrt{g}}\sum_{k=1}^{j}\alpha_k} \prod_{u<r=1}^{2j} \left| \frac{2\pi v}{L}(\tau_u - \tau_r) \right|^{2g\alpha_u\alpha_r} \quad .(70)$$

At a given order $2j$, and for $\Lambda \to \infty$, the most diverging contributions come from integrals containing an equal number of positive and negative α's. Thus, Λ-scaling of the integrals appearing in Eq.(70) is taken into account by means of a multiplicative renormalization of the effective coupling strength

$$\bar{E}_W \to \bar{E}_W(\Lambda) = \Lambda^{1-g}\bar{E}_W(\Lambda = 1) \quad . \tag{71}$$

Eq.(71) implies that the boundary interaction at $x = 0$ is irrelevant and the Neumann fixed point is always stable for $g > 1$ (i.e., in the repulsive Tomonaga-Luttinger phase). The RTL phase is always associated to a stable Neumann fixed point.

To evaluate $I(\varphi)$, one may retain only the first order terms in the E_W-expansion in Eq.(66), getting

$$\frac{Z[E_W]}{Z[0]} \approx 1 - \frac{\bar{E}_W}{2} \int_0^\beta d\tau \langle e^{i\varphi} : e^{i\sqrt{\frac{g}{2}}\chi(\tau)} : + e^{-i\varphi} : e^{-i\sqrt{\frac{g}{2}}\chi(\tau)} : \rangle \approx e^{-\beta(aE_W)\left(\frac{2\pi a}{L}\right)^g \cos(\varphi)} \quad , (72)$$

---

[3] In Eq.(68), the cutoff a has been introduced to regularize possible short-distance divergencies in the argument of the integral. It should be identified with the lattice step introduced in Eq.(2).

from which the network energy is derived as

$$E_{JJ}[\varphi] = -\lim_{\beta \to \infty} \frac{1}{\beta} \ln \frac{Z[E_W]}{Z[0]} = (aE_W) \left(\frac{2\pi a}{L}\right)^g \cos(\varphi) \quad . \tag{73}$$

Using Eq.(58) one gets $I(\varphi) \propto \sin(\varphi)$.

It is comforting to see that Eq.(73) reproduces the pertinent renormalization of the effective coupling constant given in [14].

Case B: $E_W \to 1$

When analyzing the case in which the effective coupling grows, as the size of the system goes large, we must consider that our analysis started from expanding fermionic fields whose band energy is equal to J. Accordingly, the scaling should stop as $\bar{E}_W \sim J$. The scale at which this happens, Λ^*, is found by the condition

$$\bar{E}_W(\Lambda = 1)(\Lambda^*)^{1-g} = J \tag{74}$$

This implies that scaling stops as the size of the system becomes of order of L^*, given by

$$L^* = L \left(\frac{J}{\bar{E}_W(\Lambda = 1)}\right)^{\frac{1}{1-g}} \tag{75}$$

For $L < L^*$, the theory is still perturbative. As $L \sim L^*$, instead, the system enters the nonperturbative region. In this limit, the field $\phi_L - \phi_R$ has to satisfy Dirichlet-like boundary conditions both at $x = L$ and at $x = 0$. From Eq.(57), one gets

$$\sin[\sqrt{\frac{g}{2}}[\phi_L(0 - vt) - \phi_R(0 + vt)]] = -\frac{1}{\sqrt{g}\bar{E}_W v} \frac{\partial}{\partial x}[\phi_L(0 - vt) - \phi_R(0 + vt)] \quad ; \tag{76}$$

thus, the Dirichlet-like boundary condition at $x = 0$ is

$$\phi_L(0 - vt) - \phi_R(0 + vt) = 0 \quad . \tag{77}$$

Using the mode expansion given in Eq.(A.5), Eq.(77) may be easily satisfied by setting

$$q_L = q_R \equiv q \ ; \ p_L = p_R \equiv p$$

$$\phi_L(n)e^{ik_n L} + \phi_R(-n)e^{-ik_n L} = 0 \quad , \tag{78}$$

provided that:

$$k_n = \frac{2\pi n}{L} \; ; \; n \in Z \; ; \; -\sqrt{g}4\pi p = 2\pi k + \varphi$$

$$\phi_L(n) = -\phi_R(-n) = \alpha(n) \quad . \tag{79}$$

The partition function is now given by

$$Z = \text{Tr} \exp\left[-\beta\frac{\pi v}{L}(p_L^2 + p_R^2) - \beta\frac{2\pi v}{L}\sum_{n>0}(\phi_L(-n)\phi_L(n) + \phi_R(n)\phi_R(-n))\right] \quad , \tag{80}$$

with the pertinent boundary conditions given by Eq.(79).

The trace in Eq.(80) may be factorized into a contribution from the oscillatory modes, and a contribution from the zero modes, so that

$$Z = Z_{\text{osc}} Z_{0-\text{modes}} \quad , \tag{81}$$

with $Z_{\text{osc}} = [\prod_{n=1}^{\infty}[1 - (e^{-\beta\frac{4\pi v}{L}})^n]]^{-1}$, and

$$Z_{0-\text{modes}} = \sum_{k=-\infty}^{\infty} \exp\left[-\beta\frac{v\pi}{2gL}\left(k - \frac{\varphi}{2\pi}\right)^2\right] \quad . \tag{82}$$

¿From Eqs.(81,82), one gets $I(\varphi) \propto \varphi$, for $-\pi \leq \varphi \leq \pi$; this yields the well-known sawtooth behavior [20].

The switch to this behavior from the sinusoidal behavior obtained for $E_W \sim 0$ signals the crossover from a perturbative to a nonperturbative regime of the chain. It should be observed that, for $g > 1$, the DC Josephson current has always a sinusoidal dependence on the phase difference between the bulk superconductors since, in this region, the boundary term is an irrelevant operator and, thus, the chain's behavior is always bulk-dominated.

7 Concluding remarks

Our analysis shows how a two-boundary Sine-Gordon model emerges as a pertinent effective description of a finite JJ-chain. Since our analysys heavily relies on the bosonization method, we expect that boundary Sine-Gordon models may be useful where the Tomonaga-Luttinger liquid paradigm is relevant. For instance, a magnetic spin system with a pertinent impurity at its center and with the spins at its extrema held fixed, may support a spin current across the chain, with different behaviors, depending on the boundary conditions around the impurity. Similarly, one may envisage other applications of our results to quantum wires [22] and carbon nanotubes [23].

According to the g-theorem [13], the boundary entropy of the chain should always decrease, as one gets towards the thermodynamic limit. Thus, for $L \geq L^* \to \infty$, one has

$$S_D - S_N = \lim_{\beta \to \infty} \left[\lim_{L \to \infty} \left[\ln Z_D - \ln Z_N \right] \right] \quad , \tag{83}$$

where $Z_{D/N}$ is the partition function computed with Dirichlet / Neumann boundary conditions, respectively. Eq.(83) yields a nonvanishing result only because of the contribution of the zero modes; namely, from Eq.(82), one gets that

$$S_D - S_N = \ln[\sqrt{g}] \quad . \tag{84}$$

Remarkably, the entropy variation depends on the sign of $\ln \sqrt{g}$. Thus, as $g > 1$ (i.e., within the RTLL phase) the Dirichlet boundary entropy is higher than the Neumann boundary entropy and then, in the thermodynamic limit the system flows from the Dirichlet to the Neumann fixed point. Conversely, as $g < 1$ (i.e., within the superconducting region) the Neumann fixed point carries an entropy that is higher than the one associated to the Dirichlet fixed point. Accordingly, the flow now goes from the Neumann to the Dirichlet fixed point.

The evaluation of the Josephson current from the partition function of the two-boundary Sine-Gordon model explicitly shows, for $g < 1$, a crossover from a perturbative regime ($\bar{E}_W \sim 0$), in which the current is a sinusoidal function of the phase difference at the boundary, φ, to a nonperturbative regime ($\bar{E}_W/J \geq 1$), where it exhibits a sawtooth functional dependence on φ.

There is a striking, and yet intuitive, similarity between the finite JJ-chain investigated in this paper, and an rf-SQUID in an external magnetic field [20].

For the latter system, a variation of the flux threaded by a superconducting loop operates the crossover in the behavior of the Josephson current, as a function of the applied flux. This might suggest that other very interesting condensed matter realization of boundary Sine-Gordon models, may be provided by superconducting loops, interrupted by two, or more, Josephson junctions [24]. Remarkably, using the results in Section 6, one immediately sees that the effective potential of the finite JJ-chain, as a function of the phase difference at the boundary of the weak link, i.e., $\psi = \langle \phi_L(0) - \phi_R(0) \rangle$, exhibits only one minimum within the RTLL phase (i.e. $\psi = \varphi$), since $\bar{E}_W \sim 0$, while it is a two-level quantum system in the superconducting region ($\bar{E}_W / J \geq 1$), provided that $\varphi \sim \pi$.

A   Bosonization rules

In this Appendix the bosonization rules used in this paper are reviewed.

It is a peculiar property of 1+1 dimensional theories that it is possible to realize chiral fermionic fields in terms of chiral bosonic fields, and vice versa. If one starts, for instance, from the chiral components of a free, massless, Klein-Gordon field Φ in 1+1 dimensions, the equation of motion for Φ is

$$\left[\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} \right] \Phi(x,t) = 0 \quad , \tag{A.1}$$

where periodic boundary conditions are assumed. Φ may be written, as the sum of two chiral fields as

$$\Phi(x,t) = \frac{1}{\sqrt{2}} [\phi_L(x - vt) + \phi_R(x + vt)] \tag{A.2}$$

with ϕ_L and ϕ_R chiral Fubini-Veneziano fields [25], whose mode-expansion is given by

$$\phi_L(x,t) = \phi_L(x - vt) = q_L - \frac{2\pi}{L} p_L(x - v_F t) + \frac{2\pi i}{L} \sum_{k_n} \frac{e^{ik_n(x - vt)}}{k_n} \phi_L(n) \tag{A.3}$$

and:

$$\phi_R(x,t) = \phi_R(x+vt) = q_R + \frac{2\pi}{L}p_R(x+vt) + \frac{2\pi i}{L}\sum_{k_n}\frac{e^{ik_n(x+vt)}}{k_n}\phi_R(n) \quad .(A.4)$$

The basic commutation rules are

$$[q_L,p_L] = [q_R,p_R] = i \quad ; \quad [\phi_L(n),\phi_L(m)] = n\delta_{n+m,0} \quad ; \quad [\phi_R(n),\phi_R(m)] = -n\delta_{n+m,0} \quad ,(A.5)$$

and $\{k_n\}$ is a (discrete) set of nonzero modes depending on the boundary conditions imposed on the bosonic fields (for instance, for periodic boundary conditions, one gets $k_n = \frac{2\pi n}{L}$, $n \in Z$).

Due to the commutation rules in Eq.(A.5), the bosonic vacuum $|\text{Bos}\rangle$ is defined by

$$p_L|\text{Bos}\rangle = p_R|\text{Bos}\rangle = 0$$

$$\phi_L(n)|\text{Bos}\rangle = \phi_R(-n)|\text{Bos}\rangle = 0 \quad (n>0) \quad , \tag{A.6}$$

and one may then define a creation and an annihilation part for each field, i.e.,

$$\phi_L^+(x) = q_L + \frac{2\pi i}{L}\sum_{k_n<0}\frac{e^{ik_n x}}{k_n}\phi_L(n) \quad ; \quad \phi_L^-(x) = -\frac{2\pi}{L}xp_L + \frac{2\pi i}{L}\sum_{k_n>0}\frac{e^{ik_n x}}{k_n}\phi_L(n)(A.7)$$

and

$$\phi_R^+(x) = q_R + \frac{2\pi i}{L}\sum_{k_n>0}\frac{e^{ik_n x}}{k_n}\phi_R(n) \quad ; \quad \phi_R^-(x) = \frac{2\pi}{L}xp_R + \frac{2\pi i}{L}\sum_{k_n<0}\frac{e^{ik_n x}}{k_n}\phi_R(n) \quad .(A.8)$$

¿From the commutators given in Eqs.(A.5), one sees that the modes of the operator $\frac{1}{2\pi}\frac{\partial\phi_L}{\partial x}$ obey the same algebra as the modes of the fermionic density operator, normal ordered with respect to $|\text{FS}\rangle$. Thus, the fermionic bilinear density operator $: \psi_L^\dagger\psi_L :$ may be identified with the bosonic density operator, $\frac{1}{2\pi}\frac{\partial\phi_L}{\partial x}$, provided that $|\text{Bos}\rangle$ is identified with $|\text{FS}\rangle$. The same identification may be carried for the R-modes. Therefore, one gets a first bosonization rule

$$: \psi_L^\dagger(x-vt)\psi_L(x-vt) :\rightarrow \frac{1}{2\pi}\frac{\partial\phi_L(x-vt)}{\partial x}$$

$$: \psi_R^\dagger(x+vt)\psi_R(x+vt) :\rightarrow \frac{1}{2\pi}\frac{\partial\phi_R(x+vt)}{\partial x} \quad . \tag{A.9}$$

The second rule is obtained if one identifies the chiral fermionic fields with normal ordered vertex operators of bosonic fields, defined by

$$: e^{i\alpha\phi_{L/R}(x\mp vt)} := e^{i\alpha[\phi^+_{L/R}(x\mp vt)+\phi^-_{L/R}(x\mp vt)]} e^{-\frac{\alpha^2}{2}[\phi^+_{L/R}(x\mp vt),\phi^-_{L/R}(x\mp vt)]}$$

$$= e^{i\alpha[\phi^+_{L/R}(x\mp vt)+\phi^-_{L/R}(x\mp vt)]} e^{\mp\frac{i\alpha^2\pi}{L}(x\mp vt)}\left(\frac{L}{2\pi a}\right)^{\frac{\alpha^2}{2}} \qquad . \tag{A.10}$$

The correspondence rules are now given by

$$\psi^\dagger_L(x-vt) \rightarrow \frac{1}{\sqrt{L}}: e^{i\phi_L(x-vt)}: \; ; \; \psi^\dagger_R(x+vt) \rightarrow \frac{1}{\sqrt{L}}: e^{-i\phi_R(x+vt)}: \quad .\tag{A.11}$$

To check the consistency of Eq.(A.11), one has to consider the "braiding rule":

$$: e^{i\alpha\phi_{L/R}(x)} :: e^{i\beta\phi_{L/R}(y)} := e^{i\pi\alpha\beta}: e^{i\beta\phi_{L/R}(y)} :: e^{i\alpha\phi_{L/R}(x)} : \quad , \tag{A.12}$$

and the vertex-vertex correlators:

$$\langle \text{Bos}| : e^{i\alpha\phi_{L/R}(x\mp vt)} :: e^{-i\alpha\phi_{L/R}(x'\mp vt')} : |\text{Bos}\rangle = \left[\left(\frac{\pm i}{2}\right)\frac{1}{\sin[\frac{2\pi}{L}(x-x'\mp v(t-t'))]}\right]^{\alpha^2} .\tag{A.13}$$

¿From Eqs.(A.12,A.13), one derives the basic anticommutators:

$$\{\psi_L(x-vt),\psi^\dagger_L(x'-vt')\} = \delta[x-x'-v(t-t')]$$

and

$$\{\psi_R(x+vt),\psi^\dagger_R(x'+vt')\} = \delta[x-x'+v(t-t')] \tag{A.14}$$

The other correlators used in the paper are derived from Eq.(A.13) and from Wick's theorem applied to normal ordered vertices [25].

B Renormalization Group equations for the JJ-chain with a weak link

In this Appendix, the flow of $\bar{E}_W$ is derived within the Coulomb Gas renormalizion Group approach.

To analyze the Renormalization Group flow for the JJ-chain with a weak link, one needs to observe that, at order $2j$, the short-distance most diverging contribution to the partition function is given by:

$$(\bar{E}_W)^{2j} \left(\frac{2\pi a}{L}\right)^{2jg} \int_0^\beta d\tau_1 \int_0^{\tau_1 - a/v} d\tau_2 \ldots \int_0^{\tau_{2j-1} - a/v} d\tau_{2j}$$

$$\sum_{\alpha_1 + \ldots + \alpha_{2j} = 0} \prod_{u < r = 1}^{2j} [1 - e^{-\frac{-2\pi v}{L}|\tau_u - \tau_r|}]^{2g\alpha_u \alpha_r} \tag{B.1}$$

Rescaling the short-distance cutoff as $a \to a/\Lambda$ and sending $\Lambda \to \infty$ implies the following renormalization group scaling equation for $\bar{E}_W$:

$$\frac{d\bar{E}_W(\Lambda)}{d \ln \Lambda} = (1 - g)\bar{E}_W(\Lambda) \ . \tag{B.2}$$

¿From Eq.(B.2), one may easily identify the cutoff scale $\Lambda^* = \left(\frac{J}{E_W(\Lambda=1)}\right)^{\frac{1}{1-g}}$, at which $\bar{E}_W$ becomes $\sim J$. It means that a system of size L, with a weak link of nominal strength $E_W = E_W(\Lambda = 1)$, "crosses over" towards the strongly coupled regime, as its size is increased to $L^* = \Lambda^* L$ [14].

In addition, it has to be noticed that, to remove the cutoff, one needs a further renormalization, due to "one-dimensional charge annihilation processes". This may be evidenced, for instance, by applying Anderson-Yuval-Hamann analysis of a one-dimensional instanton gas [21].

As the cutoff is rescaled from a to a/Λ, two charges, of opposite sign, may annihilate with each other, if they were originally separated by a distance between $a/(v\Lambda)$ and a/v. As a result, the integral at order $2j + 2$ should provide an extra contribution to the integral at order $2j$, which we are now going to calculate.

Upon defining

$$T = \frac{\tau_+ + \tau_-}{2} \ ; \ \tau = \tau_+ - \tau_- \tag{B.3}$$

where τ_+ being the coordinate of the $+1$-charge, and τ_- the coordinate of the -1-charge, the extra contribution arising to order $2j$, is given by

$$(\bar{E}_W)^{2j+2} \int_{a/(\Lambda v)}^{a/v} d\tau \left\{ \int_0^\beta d\tau_1 \int_0^{\tau_1 - a/L} d\tau_2 \ldots \int_0^{\tau_{2j-1} - a/v} d\tau_{2j} \times \left[\int_0^{\tau_{2j}} dT + \int_{\tau_{2j-2} - a/v}^{\tau_{2j-1} - a/v} dT + \ldots \right] \right\} \times$$

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$$\frac{1}{\left[\frac{2\pi v\tau}{L}\right]^{2g}} \prod_{u<r=1}^{2j} [1 - e^{-\frac{2\pi v}{L}|\tau_u - \tau_r|}]^{2g\alpha_u\alpha_r} \exp\left[2\tau g \sum_{k=1}^{2j} \alpha_k \frac{\partial}{\partial T} \ln[1 - e^{-\frac{2\pi v}{L}|T - \tau_k|}]\right]$$

$$+(\bar{E}_W)^{2j+2} \int_{a/(\Lambda v)}^{a/v} d\tau \left\{ \int_0^\beta d\tau_1 \int_0^{\tau_1 - a/L} d\tau_2 \ldots \int_0^{\tau_{2j-1}-a/v} d\tau_{2j} \left[\int_{\tau_{2j}-a/v}^{\tau_{2j-1}-a/v} dT + \int_{\tau_{2j-3}-a/v}^{\tau_{2j-2}-a/v} dT + \ldots \right] \right\} \times$$

$$\frac{1}{\left[\frac{2\pi v\tau}{L}\right]^{2g}} \prod_{u<r=1}^{2j} [1 - e^{-\frac{2\pi v}{L}|\tau_u - \tau_r|}]^{2g\alpha_u\alpha_r} \exp\left[2\tau g \sum_{k=1}^{2j} \alpha_k \frac{\partial}{\partial T} \ln[1 - e^{-\frac{2\pi v}{L}|T - \tau_k|}]\right] \quad .(\text{B}.4)$$

If one expands the exponentials as

$$\exp\left[\pm 2\tau g \sum_{k=1}^{2j} \alpha_k \frac{\partial}{\partial T} \ln[1 - e^{-\frac{2\pi v}{L}|T - \tau_k|}]\right] \approx 1 \pm 2\tau g \sum_{k=1}^{2j} \alpha_k \frac{\partial}{\partial T} \ln[1 - e^{-\frac{2\pi v}{L}|T - \tau_k|}] \quad ,(\text{B}.5)$$

one may derive the renormalization group equation for g [4]. The result is [21]:

$$g \to g + dg = g - \left(\frac{L}{2\pi va}\right)^g g[\bar{E}_W]^2 d\ln\Lambda \quad . \tag{B.6}$$

As expected [19], the wavefunction renormalization is needed only for $g \leq 1$.

This completes the renormalization scheme derived within the perturbative approach for the system in the presence of a weak link.

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